On exact solutions of nonlinear integrable equations via integral linearising transforms and generalised Backlund-Darboux transformations

1990 J. Phys. A: Math. Gen. 23 3761

(http://iopscience.iop.org/0305-4470/23/16/023)

J. Phys. A: Math. Gen. 23 (1990) 3761-3768. Printed in the UK

On exact solutions of nonlinear integrable equations via integral linearising transforms and generalised Bäcklund–Darboux transformations

B G Konopelchenko†

University of Paderborn, 4790 Paderborn, Federal Republic of Germany

Received 6 March 1990

Abstract. A new approach for the constructing the exact solutions of nonlinear equations via integral transforms which convert them into their linear limits is discussed. Generalised Bäcklund-Darboux transformations are introduced.

Nonlinear evolution equations integrable by the inverse spectral transform (IST) method form a broad class of partial differential equations for which one is able to construct the infinite sets of the explicit exact solutions (exactons) of different types [1-4]. Most of the methods of constructing such exact solutions (i.e. solitons, lumps, instantons, boomerons, dromions etc) are based on the use of certain auxiliary linear equations [1-4]. Wide classes of exactons can also be constructed by the Bäcklund and Darboux transformations [5-8].

In the present paper we propose a new approach for constructing the exact solutions of nonlinear equations which is a version of the general linearisation idea and based on the reinterpretation of the known formulae from the IST method. The key point is the fact that the Fourier transforms $\tilde{S}$ of the inverse problem data, which are the certain integral transforms of the potentials q, obey the linear limits of the initial nonlinear evolution equations. The exact solutions of these linear equations can be easily found by the linear Bäcklund or Darboux transformations. Then the solutions q of the nonlinear evolution equations are constructed by the usual or generalised Darboux transformations.

The main problem of this approach is the existence of the appropriate linearising integral transform. For known soliton equations the existence of such transforms is connected with the existence of the standard for the IST method auxiliary linear systems.

Here we will consider two nonlinear integrable equations, namely the nonlinear Schrödinger (NLS) equation and the Davey-Stewartson (DS) equation, as the illustration.

The NLS equation without reduction looks like [1-4]:

$$\begin{aligned} &\mathrm{i}q_t + q_{xx} + 2qrq = 0 \\ &\mathrm{i}r_t - r_{xx} - 2qrr = 0. \end{aligned} \tag{1}$$

† Permanent address: Institute of Nuclear Physics, Novosibirsk-90, 630090, USSR.

The NLS equation itself, $iq_t + q_{xx} + 2|q|^2 q = 0$, arises as the reduction $r = \bar{q}$ of the system (1). The system (1) is equivalent to the compatibility condition of the auxiliary linear system [1, 2]

$$\begin{aligned} &(-\sigma_3\partial_x + P + i\lambda)\psi = 0 \\ &(i\partial_t + 2\sigma_3\partial_x^2 - 2P\partial_x - P_x + qr\sigma_3)\psi = 0 \end{aligned} \tag{2}$$

where $P = \binom{0q}{r0}$, $\sigma_3 = \binom{10}{0-1}$ and ψ is the 2×2 matrix. The inverse problem data for (2) (in fact, part of them) are defined by the expressions [1, 2]

$$\begin{aligned} S_{12}(\lambda, t) &= \int_{-\infty}^{+\infty} dx\, e^{-2i\lambda x} q(x, t)\chi_{22}^{-}(x, t; \lambda) \\ S_{21}(\lambda, t) &= \int_{-\infty}^{+\infty} dx\, e^{2i\lambda x} r(x, t)\chi_{11}^{-}(x, t, \lambda) \end{aligned} \tag{3}$$

where $\chi^{\pm} \doteq \psi^{\pm} e^{-i\lambda\sigma_3 x}$ and $\psi^{\pm}(x, t, \lambda)$ are the fundamental matrix solutions of the system (2) with the asymptotics $\psi^{\pm}(x, t, \lambda) \to_{x\to\pm\infty} \exp(i\lambda\sigma_3 x)$. The quantities S_{12} and S_{21} obey the linear evolution equations [1, 2]

$$i\partial_t S_{12} - 4\lambda^2 S_{12} = 0 \qquad i\partial_t S_{21} + 4\lambda^2 S_{21} = 0. \tag{4}$$

Now let us introduce their Fourier transforms

$$\begin{aligned} S(x, t) &\doteq \frac{1}{\pi}\int_{-\infty}^{+\infty} d\lambda\, e^{2i\lambda x} S_{12}(\lambda, t) \\ &= \frac{1}{\pi}\int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} d\lambda\, e^{i2\lambda(x-x')} q(x', t)\chi_{22}^{-}(x', t, \lambda) \\ T(x, t) &\doteq \frac{1}{\pi}\int_{-\infty}^{+\infty} d\lambda\, e^{-2i\lambda x} S_{21}(\lambda, t) \\ &= \frac{1}{\pi}\int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} d\lambda\, e^{-i2\lambda(x-x')} r(x', t)\chi_{11}^{-}(x', t, \lambda). \end{aligned} \tag{5}$$

The functions $S(x, t)$ and $T(x, t)$ obey the linear equations

$$iS_t + S_{xx} = 0 \qquad iT_t - T_{xx} = 0. \tag{6}$$

which obviously coincide with the linear limit of equations (1). We emphasise that equations (6) can be derived directly from the definitions (5) with the use of the nonlinear system (1) and auxiliary linear system (2).

The standard reconstruction formula, for instance for q, written in terms of S is of the form [1, 2]

$$q(x, t) = \frac{1}{\pi}\int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} d\lambda\, e^{2i\lambda(x-x')} S(x', t)\chi_{11}^{-}(x, t; \lambda) \tag{7}$$

so any solution of the system (6) gives rise to the corresponding solution q, r of the nonlinear system (1).

Now let us inverse this procedure. Let us introduce the functions S and T given by the ansatz (5) (or (7)), where λ is a parameter and ψ is some matrix function to the nonlinear system (1). Further, we demand that these functions S and T should obey the linear equations (6). Then it is not difficult to check that one achieves this goal if the function ψ obeys the linear system (2). So, we arrive at the auxiliary spectral problem for equation (1). A crucial point for the applicability of such approach is, of course, the choice of the correct ansatz (5).

The general solutions of equations (6) are, of course, the superpositions of the plane waves. We restrict ourselves by the discrete superpositions

$$S(x,t)=\sum_{k=1}^{n} A_k \exp(-4i\lambda_k^2 t+2i\lambda_k x)$$
$$T(x,t)=\sum_{k=1}^{n} B_k \exp(4i\mu_k^2 t-2i\mu_k x). \tag{8}$$

where λ_k, μ_k, A_k and B_k are arbitrary complex constants.

Let us introduce the following linear Bäcklund transformations (BTs)

$$B_{\lambda k}^{(1)}: \begin{cases} B_{\lambda k}^{(1)} S = S + A_k \exp(-4i\lambda_k^2 t + 2i\lambda_k x) \\ B_{\lambda k}^{(1)} T = T \end{cases} \tag{9}$$

$$B_{\mu k}^{(2)}: \begin{cases} B_{\mu}^{(2)} S = S \\ B_{\mu k}^{(2)} T = T + B_k \exp(4i\mu^2 kt - 2i\mu_k x). \end{cases} \tag{10}$$

Using such elementary BTs, one can represents the solutions (8) as

$$S=\prod_{k=1}^{n} B_{\lambda k}^{(1)} S_0 \qquad T=\prod_{k=1}^{m} B_{\mu k}^{(2)} T_0 \tag{11}$$

where $S_0 \equiv T_0 \equiv 0$ are the trivial solutions of equations (6).

It follows from the formulae (5) that the action of the elementary BT $B_{\lambda k}^{(1)}$ is equivalent to adding the pole at the point λ_k for the quantity $S_{(12)}(\lambda, t)$. Similarity the action of $B_{\mu k}^{(2)}$ adds the pole at μ_k for $S_{(21)}(\lambda, t)$. This indicates that the transformations (9) and (10) arę nothing but the elementary BTs introduced earlier in [9]. Indeed, using the definitions (5) and the linear problem (2), one can show that the actions of the elementary BTs $B^{(1)}$(9) and $B^{(2)}$(10) are equivalent to the following gauge (Darboux) transformations

$$\psi \to^{B(1,2)} \psi' = D^{(1,2)}\psi \tag{12}$$

and

$$L_i' D^{(1,2)} = D^{(1,2)} L_i \qquad i=1,2 \tag{13}$$

where

$$D^{(1)}=\begin{pmatrix} \partial_x & 0 \\ 0 & 0 \end{pmatrix}+A \qquad D^{(2)}=\begin{pmatrix} 0 & 0 \\ 0 & \partial x \end{pmatrix}+B \tag{14}$$

and A and B are certain functions on P and P' (see [10]). In [10] it has been shown that the gauge transformations (12)-(14) give rise to the elementary BTs introduced in [9].

A very simple nonlinear superposition formula (in terms of q and r) corresponds to the elementary BTs [9]. This allows us to construct explicitly the infinite lattice of solutions $P_{n,m}=\Pi_{k=1}^{n} B_{\lambda k}^{(1)} \Pi_{k=1}^{m} B_{\mu k}^{(2)} \cdot P_0$ where $P_0=\left(\begin{smallmatrix}00\\00\end{smallmatrix}\right)$ of the system (1) [9]. Note that in the case $r=\bar{q}$ one has $T=\bar{S}$, $S_0 B_{\lambda_k}^{(2)}=\bar{B}_{\lambda k}^{(1)}$ and hence only the simultaneous action of $B_{\lambda k}^{(1)}$ and $B_{\lambda k}^{(2)}$ is admitted.

We see that the action of the elementary BTs $B^{(1)}$ and $B^{(2)}$ is of the extremely simple form in terms of the variables S and T. The possibility to convert them into explicit Bäcklund-Darboux (BD) transformations (12)-(14) is again connected with the existence of the adequate integral linearising transform (5), (7).

Note that one can consider also the other base in the manifolds of solutions of equations (6). The elementary BTs can be defined, similar to (9), (10) as the additions of the elements of the basis. A problem is to convert them into the more or less explicit forms in terms of q and r.

In conclusion, we attract attention to the fact that the integral linearising transforms (5), (7) can be rewritten in the more compact and transparent form

$$S(x,t)=\int_{-\infty}^{+\infty}\mathrm{d}x\, q(x',t)\varphi_{22}(x',t;x-x') \tag{15}$$

$$q(x,t)=\int_{-\infty}^{+\infty}\mathrm{d}x'\, S(x',t)\varphi_{11}(x,t,x-x') \tag{16}$$

where

$$\varphi(x,t;z)\doteq\frac{1}{\pi}\int_{-\infty}^{+\infty}\mathrm{d}\lambda\, \mathrm{e}^{2\mathrm{i}\lambda z}\chi^{-}(x,t,\lambda). \tag{17}$$

The function φ obeys the following linear multidimensional system of equations

$$\begin{gathered}\varphi_x-\tfrac{1}{2}[\sigma_3,\varphi_z]-\sigma_3P(x,t)\varphi=0\\ \mathrm{i}\varphi_t+2\sigma_3\varphi_{xx}-2\sigma_3\varphi_{xz}\sigma_3-\tfrac{1}{2}\sigma_3\varphi_{zz}-2P\varphi_x-P\varphi_z\sigma_3+(qr\sigma_3-P_x)\varphi=0.\end{gathered} \tag{18}$$

We emphasise that the potential P in (18) does not depend on the axuiliary variable Z. The compatibility condition for the system (18) is, of course, equivalent to the NLS system (1).

The Davey-Stewartson (DS-I) equation with the non-trivial boundaries and under the reduction $r=\bar{q}$ looks like (see e.g. [11-13]):

$$\mathrm{i}q_t+q_{\xi\xi}+q_{\eta\eta}+2\left(\int_{-\infty}^{\xi}\mathrm{d}\xi'(|q|^2)_\eta+\int_{-\infty}^{\eta}\mathrm{d}\eta'(|q|^2)_\xi\right)q+(U_1(\eta,t)+U_2(\xi,t))q=0 \tag{19}$$

where the boundary values U_1 and U_2 are arbitrary functions. The DS equation is the compatibility condition for the system

$$\begin{gathered}\begin{pmatrix}\partial_\xi & q\\ -\bar{q} & \partial_\eta\end{pmatrix}\psi=0\\ \left(\partial_t-\mathrm{i}\sigma_3(\partial_\xi-\partial_\eta)^2+\mathrm{i}\begin{pmatrix}0 & q\\ -\bar{q} & 0\end{pmatrix}(\partial_\xi-\partial_\eta)+V\right)\psi=0\end{gathered} \tag{20}$$

where V is the 2×2 matrix which depends on q and boundaries U_1, U_2 [11-13]. The Fourier transform of the inverse problem data defined as [11-13]

$$S(\xi,\eta,t)=\frac{1}{\pi}\iint_{-\infty}^{+\infty}\mathrm{d}\xi'\,\mathrm{d}\lambda\, q(\xi',\eta,t)\chi_{22}^{-}(\xi',\eta,t;\lambda)\,\mathrm{e}^{\mathrm{i}\lambda(\xi-\xi')} \tag{21}$$

where $\chi^-\doteq\psi\exp(\mathrm{i}\lambda\begin{pmatrix}-\eta & 0\\ 0 & \xi\end{pmatrix})$ is the solution of the linear integral equation corresponding to (20). The reconstruction formula for q is of the form

$$q(\xi,\eta,t)=\iint_{-\infty}^{+\infty}\mathrm{d}\eta'\,\mathrm{d}\lambda\, S(\xi,\eta',t)\chi_{11}^{-}(\xi',\eta,t;\lambda)\,\mathrm{e}^{\mathrm{i}\lambda(\eta-\eta')}. \tag{22}$$

The quantitity $S(\xi,\eta,t)$ obeys the linear equation [11-13]

$$\mathrm{i}S_t+S_{\xi\xi}+S_{\eta\eta}+(U_1(\eta,t)+U_2(\xi,t))S=0. \tag{23}$$

Equation (23) admits the separation of variables and S can be represented in the form

$$S(\xi, \eta, t) = \sum_{n=1}^{N} \sum_{m=1}^{M} \rho_{nm} X_n(\xi, t) Y_m(\eta, t) \tag{24}$$

where $\rho_{nm} \in \mathbb{C}$ and

$$\begin{aligned} &\mathrm{i} X_{nt} + X_{n\xi\xi} + U_2(\xi, t) X_n = 0 \\ &\mathrm{i} Y_{mt} + Y_{m\eta\eta} + U_1(\eta, t) Y_n = 0. \end{aligned} \tag{25}$$

The corresponding solutions of the DS-I equation are also representable in the beautiful compact form [13]

$$q(\xi, \eta, t) = 2 \sum_{n,m=1}^{N,M} Z_{nm}(\xi, \eta, t) X_n(\xi, t) Y_m(\eta, t) \tag{26}$$

where $Z = \rho(1 + \rho\beta\rho^{+}\alpha)^{-1}$, $(\rho)_{nk} = \rho_{nk}$, $(\rho^{+})_{nk} = \bar{\rho}_{kn}$ and

$$\alpha_{nk}(\xi, t) \doteq \int_{-\infty}^{\xi} \mathrm{d}\xi' \, \bar{X}_n(\xi', t) X_k(\xi', t), \; \beta_{nk}(\eta, t) = \int_{-\infty}^{\eta} \mathrm{d}\eta' \, Y_n(\eta', t) \bar{Y}_k(\eta', t). \tag{27}$$

So each pair of the explicitly solvable linear equations (25) gives rise to the explicit solution of the DS-I equation. Note that the formula similar to (26) can be derived also for the case of non-decreasing q with the use of the corresponding general non-local Riemann–Hilbert or non-local $\bar{\partial}$-problems.

In the simplest case $U_1 = U_2 \equiv 0$ the solutions X_n and Y_n of equations (25) can be chosen as the plane waves

$$X_n = \exp(-4\mathrm{i}\lambda_n^2 t + 2\mathrm{i}\lambda_k \xi) \qquad Y_n = \exp(4\mathrm{i}\mu_n^2 t + 2\mathrm{i}\mu_k \eta) \tag{28}$$

where λ_k and μ_k are arbitrary complex constants. In this case the solution (24) can be represented as follows

$$S = \prod_{n=1}^{N} \prod_{m=1}^{M} B^{(1)(\rho_{nm})}_{nm(\lambda_n,\mu_m)} S_0 \tag{29}$$

where $S_0 \equiv 0$ and BT $B^{(1)(\rho_{nm})}_{nm(\lambda_n,\mu_m)}$ acts as

$$B^{(1)(\rho_{nm})}_{nm(\lambda_n,\mu_m)} \cdot S = S + \rho_{nm} X_n(\xi, t) Y_m(\eta, t). \tag{30}$$

On the submanifold of degenerate solutions ($\rho_{nm} = a_n b_m$) BT $B^{(1)(\rho_{nm})}_{nm(\lambda_n,\mu)}$ can be represented as the tensor product of the two more elementary BTs:

$$B^{(1)(a_n b_m)}_{nm(\lambda_n,\mu_m)} = B^{(1)+}_{n(\lambda_n)} \otimes B^{(1)-}_{m(\mu_m)} \tag{31}$$

where

$$\begin{aligned} &B^{(1)+}_{n(\lambda_n)} \cdot X = X + a_n X_n(\xi, t) \\ &B^{(1)-}_{m(\mu_m)} \cdot Y = Y + b_m Y_m(\eta, t). \end{aligned} \tag{32}$$

This fine structure of the elementary BTs is an important feature of the (2+1)-dimensional case.

Indeed these BTs are more elementary than those constructed in [14]. Using (29), it is not difficult to see that BT $B^1_{\lambda_n}$, introduced in [14], is

$$B^1_{\lambda_n} = \prod_{\mu=-\infty}^{+\infty} B^{(1)\rho_n(\mu)}_{n\mu(\lambda_n,\mu)}$$

where $\Pi_{\mu=-\infty}^{+\infty}$ means the continual product. This BT $B_{\lambda_n}^{(1)}$ (together with the similar BT $B_{\mu_m}^{(2)}$) generates the lattice of solutions, constructed in [14], which is the submanifold of solutions (26).

Then considering the BT $B^{(1)}$ with both continual parameters and degenerate functions ρ of the Gaussian type, one can construct the gausson type solutions of the DS-I equation, firstly found by the different approach in [15] (see also [16]).

An interesting solution of the DS-I equation with the Gaussian localisation of the different type arises in the case $U_1 = -\alpha^2(\xi - v_1 t)^2$, $U_2 = -\alpha^2(\eta - v_2 t)^2$ where v_1 and v_2 are constants. The corresponding functions X_n and Y_n are the well known eigenfunctions of the stationary states for the moving one-dimensional harmonic oscillator, (see e.g. [17]):

$$X_n(\xi, t) = \left(\frac{2\alpha^2}{\pi^2}\right)^{1/4} (2^n n!)^{-1/2} \exp[-\mathrm{i}\tfrac{1}{2}v_1 t + v_1\xi) + 2\mathrm{i}\alpha(n+\tfrac{1}{2})t] \times \exp[-\alpha^2(\xi - v_1 t)^2] H_n(\sqrt{2\alpha}(\xi - v_1 t) \tag{33}$$

where H_n are the Hermitian polynomials. The functions Y_n are also given by (33) with the substitution $\xi \to \eta$. Formula (26) gives

$$q(\xi, \xi, t) = Z(\xi, \eta, t) \exp[-\alpha^2((\xi - v_1 t)^2 + (\eta - v_2 t)^2)] \tag{34}$$

where Z is calculated by the formulae (26) and (27). Let us note that in this case we have, at $t \to \infty$, increasing moduli $|U_1|$ and $|U_2|$ of boundaries U_1 and U_2.

This exacton can be driven by changing the boundaries similar to the dromions [13]. But, in contrast to dromions, the exacton (34) does not radiate energy if its motion is not uniform.

Now let us consider the dromion solutions of the DS-I equations [12, 13, 18]. They correspond to the reflectionless potentials U_1 and U_2. As known, the reflectionless potentials and corresponding eigenfunctions can be constructed by the Darboux transformation (DT) [8]. So, if one defines the elementary DTs by the formulae

$$D_1^+: \begin{cases} X \to X' = (\partial_\xi - (\log X_1)_\xi)X \\ U_2 \to U_2' = U_2 + 2(\log X_1)_{\xi\xi} \end{cases} \tag{35}$$

$$D_2^-: \begin{cases} Y \to Y' = (\partial_\eta - (\log Y_1)_\eta)Y \\ U_1 \to U_1' = U_1 + 2(\log Y_1)_{\eta\eta} \end{cases} \tag{36}$$

where X_1 and Y_1 are some solutions of equations (25), then the general degenerate solution (24) ($\rho_{nm} = a_n b_m$) can be represented in the form

$$S_{N,M} = \prod_{K=1}^{N} D_K^+ \otimes \prod_{K=1}^{M} D_K^- \cdot S_0 \tag{37}$$

where $S_0 = X_0 \cdot Y_0$ and X_0 and Y_0 are the solutions of equations (25) with $U_1 = U_2 = 0$.

So, the dromions with the different (N, M) are connected to each other by the elementary DTs (35) and (36). Using formulae (20)-(22), one can, in principle, find the action of the elementary DTs on the initial variables ψ and q. These transformations are the Darboux-Bäcklund transformations of a new type since they change also the boundaries.

Using the known results for the DTs [8], one is also able to construct rational, finite-gap and other types of solutions for the system (25) and correspondingly for the DS-I equation.

We emphasise that the linearising transform (21), (22) again plays a key role in all the constructions discussed. Similar to the NLS equation the formulae (21) and (22) can be rewritten in a compact form

$$S(\xi, \eta, t) = \int_{-\infty}^{+\infty} \mathrm{d}\xi' \, q(\xi', \eta, t)\varphi_{22}(\xi', \eta, t, \xi - \xi') \tag{38}$$

$$q(\xi, \eta, t) = \int_{-\infty}^{+\infty} \mathrm{d}\eta' \, S(\xi', \eta, t)\varphi_{11}(\xi', \eta, t, \eta - \eta') \tag{39}$$

where $\varphi(\xi, \eta, t; z) \doteq (1/\pi) \int_{-\infty}^{+\infty} \mathrm{d}\lambda \, {}^{\mathrm{i}\lambda z}\chi^{-}(\xi, \eta, t; \lambda)$.

This function φ obeys the following multidimensional linear equation

$$\begin{pmatrix} \partial_\xi & 0 \\ 0 & \partial_\eta \end{pmatrix} \varphi - \tfrac{1}{2}[\sigma_3, \varphi_z] - \begin{pmatrix} 0 q \\ \bar{q} 0 \end{pmatrix} \varphi = 0 \tag{40}$$

plus the corresponding equation with φ_t.

A similar situation takes place for the other soliton equations. For instance, for the Korteweg-de Vries equation $U_t + U_{xxx} + 6UU_x = 0$ [1-4] the linearising integral transform looks like

$$U(x, t) = \frac{\partial}{\partial x} \int_{-\infty}^{+\infty} \mathrm{d}x' \, S(x', t)\varphi(x, t; x - x'). \tag{41}$$

Now $S_t + S_{xxx} = 0$ and $\varphi(x, t; z)$ obeys the linear system of equations

$$\begin{aligned} &\varphi_{xx} - 2\varphi_{xz} + U(x, t)\varphi = 0 \\ &\varphi_t + 4\varphi_{xxx} - 12\varphi_{xxz} + 12\varphi_{xzz} + 6U(\varphi_x - \varphi_z) + 3U_x\varphi = 0. \end{aligned} \tag{42}$$

For all soliton equations the integral linearising transforms have a form similar to (15), (16), (38), (39), (41) and the function φ obeys the linear multidimensional systems with the potential which is independent on the auxiliary variable Z. This appears as the general form of their ansatz for the linearising transform.

So, the approach under discussion can be formulated as follows. For a given nonlinear equation, introduce the appropriate integral transform (of the type (15), (16), (38), (39), (41)) which would allow us to convert it into the linear equation for S and find the corresponding system of equations for φ. In fortuitous cases this system will be the linear one (or, at least, solvable) and one will be able to convert the solutions of the linear equation for S into the solutions of the nonlinear equation for U.

Of course, for solition equations this scheme is equivalent to many previously proposed approaches. Nevertheless it is of interest since it may give us solvable nonlinear equations of a new type.

Acknowledgments

I am thankful to P Santini for the stimulating discussion. I am also very grateful to the Paderborn University and especially to B Fuchssteiner for the kind hospitality and the DFG for the support.

References

[1] Zakharov V E, Manakov S V, Novikov S P and Pitaevski L P 1980 *Theory of solitons* (Moscow: Nauka) (Engl. transl. 1984 New York: Plenum)

[2] Ablowitz M Y and Segur H 1981 *Solitons and Inverse Scattering Transform* (Philadelphia, PA: SIAM)

[3] Calogero F and Degasperis A 1982 *Spectral Transform and Solitons I* (Amsterdam: North-Holland)

[4] Newell A C 1985 *Solitons in Mathematics and Physics* (Philadelphia, PA: SIAM)

[5] Miura R M (ed) 1976 *Bäcklund Transformations, the Inverse Scattering Method, Solitons and their Applications* (*Lecture Notes in Mathematics* **515**) (Berlin: Springer)

[6] Rogers C and Shadwick W F 1982 *Bäcklund Transformations and their Applications* (New York: Academic)

[7] Konopelchenko B G 1987 *Nonlinear Integrable Equations* (*Lecture Notes in Physics* **270**) (Berlin: Springer)

[8] Matveev V B and Salle M A *Darboux Transformations and Solitons* (Berlin: Springer) in press

[9] Konopelchenko B G 1982 *Phys. Lett.* **87A** 445

[10] Boiti M, Pempinelli F and Tu G Z 1983 *Phys. Lett.* **93A** 107

[11] Fokas A S and Santini P M 1989 *Phys. Rev. Lett.* **63** 1329

[12] Fokas A S and Santini P M 1990 *Physica* D in press

[13] Santini P M and Fokas A S 1989 *preprint* 684 Rome University

[14] Boiti M, Konopelchenko B G and Pempinelli F 1985 *Inverse Problems* **1** 33

[15] Cornille H Y 1979 *Math. Phys.* **20** 199

[16] Degasperis A 1990 *Proc. Conf. Nonlinear Evolution Equations Solitons and the Inverse Scattering Transform Oberwolfach* (Singapore: World Scientific)

[17] Landau L D and Lifshitz E M 1972 *Quantum Mechanics* (Moscow: Nauka)

[18] Boiti M, Leon Y Y-P, Martina L and Pempinelli F 1988 *Phys. Lett.* **132A** 432